TCC Week 5

Always assume $-\Delta + V$ satisfies (a)

Def. We say $v \geq 0$ is a small sub-solution at infinity for $-\Delta + V$ in $\Omega = \mathbb{R}^N \setminus \bar{B}_1$ if

$\exists$ a supersolution $u_* > 0$ such that

$$\lim_{|x| \to \infty} \frac{v(x)}{u_*(x)} = 0.$$

We say $v \geq 0$ is a large sub-solution at infinity for $-\Delta + V$ in Ω if v is not a small subsolution, $\forall$ supersolution $u_* > 0$,

$$\limsup_{|x| \to \infty} \frac{v(x)}{u_*(x)} > 0$$

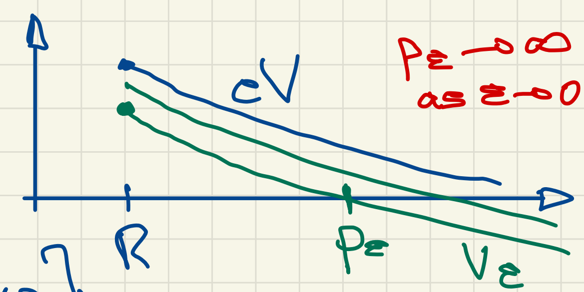
Lemma 1. If V is a small subsolution at ∞, then $\forall$ positive supersolution u,
$\exists\, c>0$: $u \geq cV$ in $\bar{B}_R^c$, for some $R>1$

◀ We may assume $\sup_{|x|=R} V = 1$. Set $c = \inf_{|x|=R} u > 0$

Consider $V_\varepsilon = cV - \varepsilon u_*$, $u_*>0$ is from the Defn. of small sub-sol.

1) $V_\varepsilon < u$ on $|x| = R$

2) $V_\varepsilon \leq 0$ on $|x| \geq \rho_\varepsilon \gg R$



$\Rightarrow$ By Comparison Pr. on $A_{R,\rho_\varepsilon} = \{R < |x| < \rho_\varepsilon\}$,

$V_\varepsilon \leq u$ in A_{R,ρ_ε} $\Longrightarrow$ $V \leq u$ in B_R^c ▶

Example: $-\Delta$ in $\mathbb{R}^N \setminus \overline{B_1}$, $N \geq 3$

$\Rightarrow$ $v_1 = |x|^{2-N}$ is a small subsolution

◀ $u_* = 1$ – supersolution, $\lim\limits_{|x|\to 0} \frac{v_1}{u_*} = 0$ ▶

Exercise: $u_{**} = |x|^{\frac{2-N}{2}}$ is a supersolution too,

$\lim\limits_{|x|\to\infty} \frac{v_1}{u_{**}} = 0$

Are there any large subsolutions?

Lemma 2. Assume that $-\Delta v + V v \le 0$ in B_1^c, $v \ge 0$, $v \not\equiv 0$, $v = 0$ on $|x| = 1$.

Then v is a large subsolution.

◀ Assume v is not large = small $\Longrightarrow$

$\exists\, u_* > 0$, $-\Delta u_* + V u_* \ge 0$ in B_1^c,

$\lim_{|x|\to\infty} \frac{v}{u_*} = 0$. As before, $v_\varepsilon := v - \varepsilon u_*$.

Then: 1) $v_\varepsilon = 0$ on $|x| = 1$

2) $v_\varepsilon \le 0$ on $|x| = \rho_\varepsilon > 1$ (same as before)

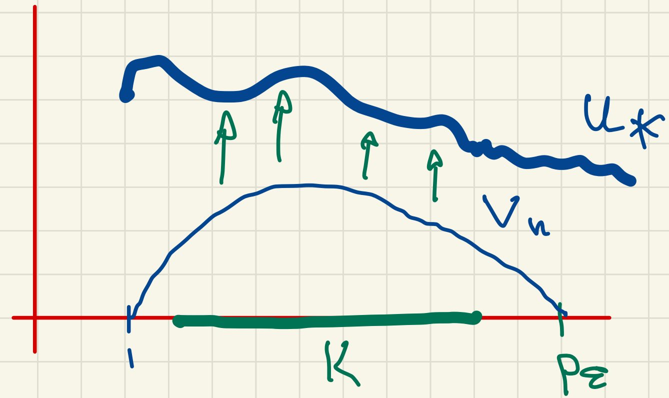
Fix a compact set $K \subset \overline{B_1}^c$.

Take $p_\varepsilon \gg 1$ so that $K \subset A_{1,p_\varepsilon}$ $\left(p_\varepsilon \xrightarrow[\varepsilon \to 0]{} \infty\right)$

Take $V_n = n V_\varepsilon$.

So $V_n \leq u_* $ on A_{1,p_ε}

because $V_n = 0$ on the Boundary.



But $V_n(x) \xrightarrow[n \to \infty]{} +\infty \quad \forall x \in K$

$\Longrightarrow u_*(x) = +\infty \ \forall x \in K$ — a contradiction!

Example: $V = 1 - |x|^{2-N}$. Then V is a large solution for $-\Delta$ in $\mathbb{R}^N \setminus \overline{B}_1$, $N \geq 3$.

◀ $-\Delta V = 0$ in B_1^c, $V = 0$ on $|x| = 1$. ▶

Exercise: $-\Delta$ in $\mathbb{R}^2 \setminus \overline{B}_1$. Then

1) $V_1 = 1$ is a small sub-sol (take $u_* = \log|x|$)

2) $V_0 = \log|x|$ is a large sub-sol

Phragmen-Lindelöf principle

Let $u > 0$ be a supersolution to $-\Delta + V$ in $\overline{B_1}^c$.

Then: 1) for any small subsolution v at ∞ $\exists R > 1$, $c > 0$: $u \geq c v$ in $\overline{B_R}^c$

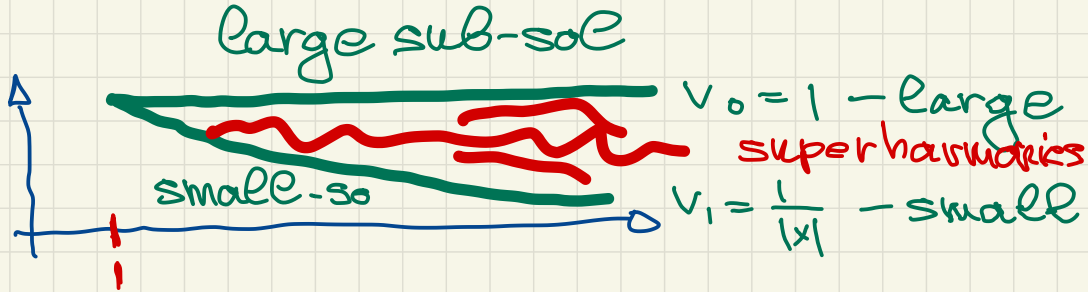
2) for any large subsolution w at ∞

$$\liminf_{|x| \to \infty} \frac{u}{w} < +\infty$$

◀ 1) = Lemma 1 ; 2) = Def. of large sub-sol. ▶

[Protter-Weinberger]

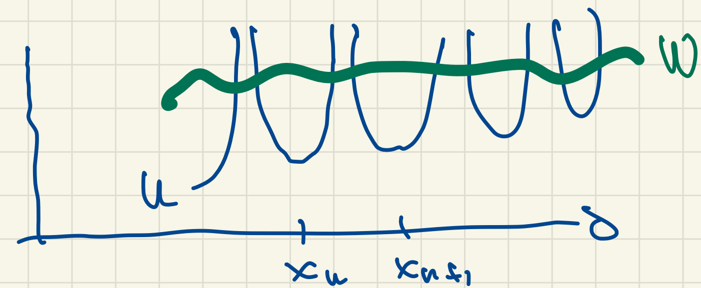
$-\Delta$ in $\mathbb{R}^3$



Remarks:

1) $-\Delta V \le 0 \implies V$ "$\le$" u (every supers) or V "$\ge$" u (every supers.)

2) $\liminf_{|x|\to\infty} \frac{u}{w} < +\infty$: (u – supersol.)



$\exists\ x_n \to 0$: $\lim_{|x|\to\infty} \frac{V(x_n)}{w(x_n)} < +\infty$

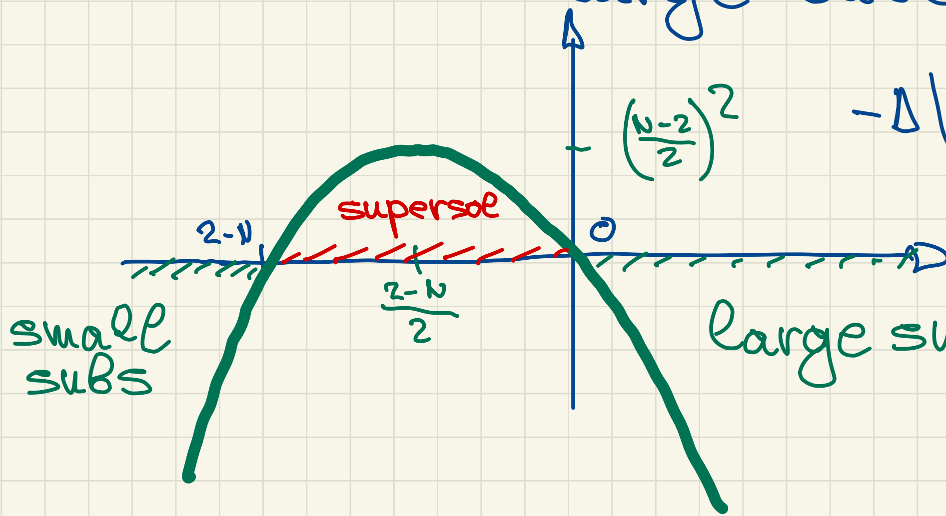
"comparison on a sequence of points"

Example: $-\Delta$ in $\mathbb{R}^N \setminus \overline{B}_1$, $N \geq 3$

$|x|^\gamma$ is a supersolution if $-(N-2) \leq \gamma \leq 0$

small subsol if $\gamma \leq -(N-2)$

large subsol if $\gamma \geq 0$



$$-\Delta |x|^\gamma = -\gamma(\gamma+N-2)|x|^{\gamma-2}$$

Exersise: $-\Delta$ on $\mathbb{R}^2 \setminus \overline{B}_1$:

$u = \log^\beta |x|$ — supersol if $\beta \in [0,1]$
small subsol if $\beta < 0$
large subsol if $\beta > 1$

$v = r^\gamma$, $\gamma \le 0$ — small subsol
$\gamma \ge 0$ — large subso

Exersise: Describe small/large subsol. to $-\Delta - \frac{c}{|x|^2}$ in $\mathbb{R}^N \setminus \overline{B}_1$ in the form $u = c|x|^\gamma$.

Example: Consider $-\Delta + V$ in $\mathbb{R}^N \setminus \overline{B}_1$,

$$\lim_{|x|\to\infty} \frac{|V(x)|}{|x|^{2+\varepsilon}} = 0 \quad \text{and } -\Delta + V \text{ satisfies } (\lambda)$$

Then $-\Delta u + Vu \geq 0$ in $\mathbb{R}^N \setminus \overline{B}_1$, $u > 0 \implies$

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|^{-(N-2)}} > 0, \quad \liminf_{|x|\to\infty} u(x) < +\infty$$

◀ Take $V_1 = |x|^{-(N-2)} + a|x|^{-(N-2)-\delta_1}$, $0<a, \delta_1$ - small

is a small subsolution $\quad -\Delta V_1 + V V_1 \leq 0$ in B_1^c

$V_0 = 1 - a|x|^{-(N-2)+\delta_2}$, $0<a, \delta_2$ - small

is a large subsolution ▶